P

Exam Introduction to Logic

January 20th, 2014

- ☞ Part A consists of 5 exercises and part B of 4 exercises.
- ☞ Philosophy students only need to do part A of the exam, other students must do both part A and part B.
- ☞ Write only your student number at the top of the exam. Also put your number at the top of any additional pages.
- ☞ Use a blue or black pen (so no pencils, red pen or marker).
- ☞ Don't forget to fill out and hand in the anonymous evaluation. (N.B. philosophy students will be given an evaluation form after finishing part S later this week.)
- ☞ Only hand in your definite answers. You can take the exam questions and any drafts home.
- ☞ When you hand in your exam, wait until the supervisors have checked whether all information is complete. They will indicate when you can leave.

Doesn't work

GOOD LUCK!

$\neg P \to W$

Part A

A1: translating propositional logic (40 points) Translate the following sentences to *propositional logic*. Atomic sentences are represented by uppercase letters. Do not forget to provide the translation key.

W P ¬P W

a. John is neither creative nor ambitious, and he only works if he is under pressure.

b. If John is criticized, he does not show emotions unless he is angry.

A2: translating first-order logic (40 points) Translate the following sentences to *first-order logic*. Do not forget to provide the translation key. The domain of discourse is the set of all humans.

a. If everyone is happy and nobody has made a compromise, then there are some broad-minded people.

b. There is someone who gives a flower to everyone who smiles at Hassan.

A3: formal proofs (51 points) Give formal proofs of the following inferences. Do not forget the justifications.

a. $A \wedge B$

$C \vee \neg A$

$(C \to \neg A) \to D$

b. $\forall x(P(x) \to Q(x))$
$\exists x \neg Q(x)$

$\neg \forall x P(x)$

c. $\exists x \forall y(P(y) \leftrightarrow x = y)$

$\exists x P(x)$

A4: truth tables (72 points) Answer the following questions using truth tables. Write down the complete truth tables and motivate your answers.

a. Check with a truth table whether the following argument is tautologically valid.

$(\neg B \wedge A) \vee (B \wedge A)$
$\neg A \vee B$

$A \leftrightarrow B$

b. Check with a truth table whether the following formulas are tautologically equivalent.

(i) $(A \vee (\neg B \to (A \wedge B))) \vee C$
(ii) $\neg(C \leftrightarrow A) \vee (\neg B \vee (\neg A \to \neg C))$

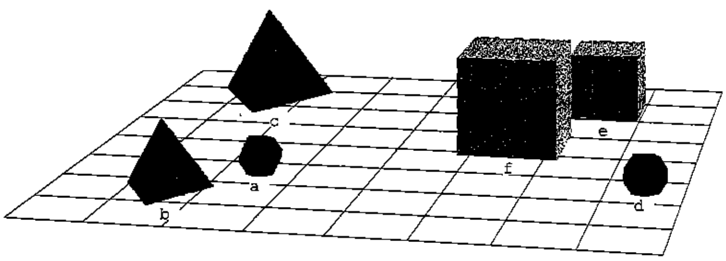
c. Check with a truth table whether the following formula is a contradiction.
$A \to (\neg B \to (A \leftrightarrow B))$

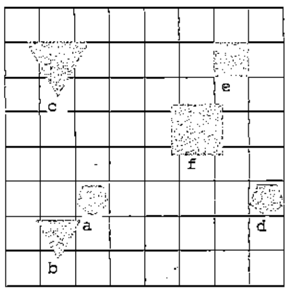
d. Check with a truth table whether the following argument logically valid. Indicate clearly which rows are spurious.

SameCol(a, a) $\wedge$ (Tet(b) $\vee$ Cube(b))

$\neg$Tet(b) $\to$ Cube(b)

A5: Tarski's World (97 points)





In the world displayed above a and d are small, c and f are large, and the other objects are medium.

a. In the world displayed above there are at least two cubes. How can you express this with one formula in the language of Tarski's World such that the formula would be true in every world with at least two cubes, and false if there were not at least two cubes?

Small a d

b. Indicate of each formula below, whether it is true or false in the world displayed above. You do not need to explain your answers.

(i) Tet(c) $\leftrightarrow$ (Large(b) $\leftrightarrow$ Medium(f)) T

(ii) RightOf(a, b) $\vee$ BackOf(c, d) T

(iii) SameSize(c, f) $\wedge$ $\neg$Between(a, e, b) T

(iv) $\neg$Small(d) $\leftrightarrow$ FrontOf(a, d) T

(v) $\forall$x($\neg$Dodec(x) $\rightarrow$ Medium(x)) F

(vi) $\exists$x(RightOf(x, a) $\wedge$ Larger(x, e) $\wedge$ Dodec(x)) F

(vii) $\forall$x((FrontOf(x, f) $\vee$ LeftOf(x, f)) $\leftrightarrow$ (x = a $\vee$ Tet(x))) F

(viii) $\exists$x(Cube(x) $\wedge$ $\forall$y(Smaller(x, y) $\vee$ x $\neq$ y)) F

(ix) $\forall$x$\forall$y($\neg$Large(x) $\rightarrow$ Larger(y, x)) F

(x) $\forall$x$\exists$y(Smaller(x, y) $\rightarrow$ $\neg$Medium(x)) T

c. Explain how the formula below can be made true by removing one object from the world displayed above.

$$\exists x \forall y ((\neg \mathsf{Tet}(y) \wedge \neg \mathsf{Cube}(y)) \leftrightarrow x = y)$$

Part B

B1: Normal forms propositional logic (20 points)

a. Provide a disjunction normal form (DNF) of the following formula. Show all of the intermediate steps.

$(A \rightarrow (B \rightarrow C)) \wedge (C \rightarrow \neg A)$

b. Provide a conjunction normal form (CNF) of the following formula. Show all of the intermediate steps.

$(\neg A \wedge B) \vee (C \rightarrow \neg D)$

B2: Normal forms first-order logic (45 points)

a. Provide a Prenex normal form of the following formula. Show all of the intermediate steps.

$\exists x P(x) \rightarrow (\forall x Q(x) \wedge \neg \exists x \neg S(x))$

b. Provide a Skolem normal form of the following formula. Show all of the intermediate steps.

$\exists x \forall y (R(x, y) \rightarrow \exists z Q(x, z, y))$

c. Check the satisfiability of the Horn sentence below using the Horn algorithm. If you prefer the conditional form, you may also use the satisfiability algorithm for conditional Horn sentences.

$$A \wedge (\neg A \vee B \vee \neg C) \wedge (\neg C \vee \neg D) \wedge (\neg A \vee E) \wedge (\neg D \vee \neg E)$$

B3: Translating function symbols (40 points) Translation the following sentences using the translation key provided. The domain of discourse is the set of all humans.

father(x): the father of x
mother(x): the mother of x

Younger(x, y): x is younger than y

jan: Jan
mieke: Mieke

a. Someone who is younger than Jan's mother is younger than Mieke's father.

b. Although Jan is younger than Mieke, the mother of Jan's father is not younger than Mieke's father.

c. Everyone who has the same mother as Mieke, has the same father as Mieke.

d. Someone who is younger than Jan is younger than both of Mieke's grandfathers.

B4: Semantics (45 points) Let a model $\mathfrak{M}$ with domain $\mathfrak{M}(\forall) = \{1, 2, 3\}$ be given such that

- $\mathfrak{M}(a) = 1$,
- $\mathfrak{M}(b) = 2$,
- $\mathfrak{M}(c) = 3$,
- $\mathfrak{M}(P) = \{1, 2\}$
- $\mathfrak{M}(Q) = \{2, 3\}$
- $\mathfrak{M}(R) = \{\langle 1, 2\rangle, \langle 3, 3\rangle\}$

1

Let h be an assignment such that $h(x) = 3$, $h(y) = 2$, en $h(z) = 1$.
Evaluate the following statements. Follow the truth definition step by step.

a. $\mathfrak{M} \models Q(c) \wedge (Q(b) \to R(y, x))[h]$

b. $\mathfrak{M} \models \exists x(P(x) \wedge Q(x))[h]$

c. $\mathfrak{M} \models \forall x \forall y(\neg Q(x) \to (R(x, y) \vee R(y, y)))[h]$

A1	A2	A3	A4	A5	B1	B2	B3	B4
40	[illegible]	51	43	81	16	30	40	45

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A1 a) 40

C: John is creative

A: John is ambitious

P: John is under pressure

20 W: John works

only if

$\neg C \wedge \neg A \wedge (W \to P)$

b)

C: John is criticized

E: John shows emotions

A: John is angry

20

$C \to (\neg E \vee A)$

A2 a)

H(x): x is happy

O(x): x has made a compromise

B(x): x is broad-minded

[illegible]

20

$\forall x (H(x) \wedge \neg O(x)) \to \exists x B(x)$

b)

F(x,y): x gives a flower to y

S(x,y): x smiles at y

20 h: Hassan

$\exists x \forall y (\cancel{F(x,y) \wedge S(x,y)} \; S(y,h) \to F(x,y))$

A3 a)

1	$A \wedge B$	
2	$C \vee \neg A$	
3	$C \rightarrow \neg A$	
4	C	
5	A	$\wedge$ Elim(1)
6	$\neg A$	$\rightarrow$ Elim(3, 4)
7	$\bot$	$\bot$ Intro(5, 6)
8	$\neg A$	
9	A	$\wedge$ Elim(1)
10	$\bot$	$\bot$ Intro(8, 9)
11	$\bot$	$\vee$ Elim(2, 4-7, 8-10)
12	D	$\bot$ Elim(11)
13	$(C \rightarrow \neg A) \rightarrow D$	$\rightarrow$ Intro(3-12)

17

b)

~~$\forall x\,(P(x) \rightarrow Q(x))$~~

~~$\exists x \neg Q(x)$~~

~~$\forall x\, P(x)$~~

~~$\boxed{c}\ \neg Q(c)$~~

~~$P(c)$~~

~~$P(c) \rightarrow Q(c)$~~

~~$Q(c)$~~

~~$\bot$~~

~~$\neg P(c)$~~

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~~A3 b) $\forall x (P(x) \to Q(x)$~~
~~$\exists \neg Q(x)$~~
~~$\boxed{c}\ \neg Q(c)$~~
~~$P(c) \to Q(c)$~~

b)
1 $\forall x (P(x) \to Q(x))$
2 $\exists x \neg Q(x)$
3 $\forall x P(x)$
4 $\boxed{c}\ \neg Q(c)$
5 $P(c)$ — $\forall$Elim(3)
6 $P(c) \to Q(c)$ — $\forall$Elim(1)
7 $Q(c)$ — $\to$Elim(5, 6)
8 $\perp$ — $\perp$ Intro(4, 7)
9 $\perp$ — $\exists$ Elim(2, ~~4~~ -8)
10 $\neg \forall x P(x)$ — $\neg$Intro(3-9)

17

c)
1 $\exists x \forall y (P(y) \leftrightarrow x = y)$
2 $\boxed{c}\ \forall y (P(y) \leftrightarrow c = y)$
3 $P(c) \leftrightarrow c = c$ — $\forall$Elim(2)
4 $c = c$ — $=$Intro
5 $P(c)$ — $\leftrightarrow$ Elim(3, 4)
6 $\exists x P(x)$ — $\exists$Intro(5)
7 $\exists x P(x)$ — $\exists$Elim(1, 2-6)

17

A4

a)

A	B	(¬B	∧	A)	∨	B ∧ A	¬A	∨ B	A ↔ B
T	T	F	F		T	T	F	T	T
T	F	T	T		T	F	F	F	F
F	T	F	F		F	F	T	T	F
F	F	T	F		F	F	T	F -3	T

9

In the last row the two premises are false but ~~that~~ $A \leftrightarrow B$ is true. Therefore the argument is not tautologically valid

-6

6

b) (ii) is a tautology while (i) is not. In the last row (i) is ~~true~~ false while (ii) is true. Therefore, (i) and (ii) are not tautologically equivalent.

A	B	C	(A ∨	(¬B	→	(A ∨ B)))	∨ C
T	T	T	T	F	T	T	T
T	T	F	T	F	T	T	T
T	F	T	T	T	T -3	T -3	T
T	F	F	T	T	T -3	T -3	T
F	T	T	T	F	T	T	T
F	T	F	T	F	T	T	T
F	F	T	F	T	F	F	T
F	F	F	F	T	F	F	F

A	B	C	¬(C ↔ A)		∨	(¬B	∨ (
T	T	T	F	T	T	F	T
T	T	F	T	F	T	F	T
T	F	T	F	T	T	T	T
T	F	F	T	F	T	T	T
F	T	T	T	F	T	F	F
F	T	F	F	T	T	F	T
F	F	T	T	F	T	T	T
F	F	F	F	T	T	T	T

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c)

A	B	$A \to (\neg B \to (A \leftrightarrow B))$			
T	T	T	F	T	T
T	F	F	T	F	F
F	T	T	F	T	F
F	F	T	T	T	T

10

The formula is not a contradiction, it is true in the first, third and fourth row

d)

	Same Col(a,a)	Tet(b)	Cube(b)	SameCol(a,a) ∧ (Tet(b)	∨ Cube(b))	¬Tet(b) → Cube(b)	
-3 *	T	T	T	T	T	F	T
	T	T	F	T	T	F	T
	T	F	T	T	T	T	T
	T	F	F	F	F	T	F
*	F	T	T	F	T	F	T
*	F	T	F	F	T	F	T
*	F	F	T	F	T	T	T
*	F	F	F	F	F	T	F

10

Rows with an asterix (*) are spurious rows as a is always in the same column ~~rows~~ as itself.

-5

Since there are no rows where the premis is ~~false~~ true and the conclusion is ~~true~~ false, the argument is logically valid ~~[illegible]~~

~~[illegible]~~

A58 a) $\exists x \exists y (x \neq y \wedge Cube(x) \wedge Cube(y))$

b)(i) True
ii) True
iii) ~~True~~
iv) ~~False~~ True
64 v) False
vi) False
vii) False
viii) False
ix) False
x) True

RightOf(x,y). x is right of y

c) Only dodecs are neither tet's nor cubes. All objects are identical to themselves and not identical to any other object

9

To make the formula true, we simply take away one of the dodecs, for example, a.

We take the remaining Dodec d, for which it is true that

$\neg Tet(d) \wedge \neg Cube(d)$,

as the x for which the formula holds

$(\neg Tet(d) \wedge \neg Cube(d)) \leftrightarrow$ [illegible] $d = d$

$(\neg Tet(b) \wedge \neg Cube(b)) \leftrightarrow$ [illegible] $d = b$ [illegible]

$(\neg Tet(c) \wedge \neg Cube(c)) \leftrightarrow$ [illegible] $d = c$ [illegible]

And so forth for all the other objects

B1

10 a) $(A \rightarrow (B \rightarrow C)) \wedge (C \rightarrow \neg A)$

$\Leftrightarrow (\neg A \rightarrow (B \rightarrow C)) \wedge (\neg C \vee \neg A)$

$\Leftrightarrow (\neg A \vee (\neg B \vee C)) \wedge (\neg C \vee \neg A)$

$\Leftrightarrow (\neg A \vee \neg B \vee C) \wedge (\neg C \vee \neg A)$

$\Leftrightarrow (\neg C \wedge (\neg A \vee \neg B \vee C)) \vee (\neg A \wedge (\neg A \vee \neg B \vee C))$

$\Leftrightarrow (\neg C \wedge \neg A) \vee (\neg C \wedge \neg B) \vee (\neg C \wedge C) \vee (\neg A \wedge \neg A) \vee (\neg A \wedge \neg B)$ [illegible]

$\Leftrightarrow (\neg C \wedge \neg A) \vee (\neg C \wedge \neg B) \vee \bot \vee (\neg A) \vee (\neg A \wedge \neg B)$ [illegible]

$\Leftrightarrow (\neg C \wedge \neg A) \vee (\neg C \wedge \neg B) \vee \neg A \vee (\neg A \wedge \neg B) \vee (\neg A \wedge C)$

6

b) $(\neg A \wedge B) \vee (C \rightarrow \neg D)$ -4

$\Leftrightarrow (\neg A \wedge B) \vee (\neg C \vee \neg D)$

$\Leftrightarrow (\neg A \wedge B) \vee \neg C \vee \neg D$

$\Leftrightarrow (\neg A \vee \neg C \vee \neg D) \wedge (B \vee \neg C \vee \neg D)$

B2

a) 15 $\exists P(x) \rightarrow (\forall x\, Q(x) \wedge \neg \exists x \neg S(x))$

$\exists P(x) \rightarrow (\forall x\, Q(x) \wedge \forall x \neg\neg S(x))$

$\Leftrightarrow \neg \exists_x P(x) \vee (\forall x\, Q(x) \wedge \forall z\, S(z))$

$\Leftrightarrow \forall x \neg P(x) \vee (\forall y\, Q(y) \wedge \forall z\, S(z))$

$\Leftrightarrow \forall x \neg P(x) \vee (\forall y \forall z (Q(y) \wedge S(z)))$

$\Leftrightarrow \forall x \forall y \forall z (\neg P(x) \vee (Q(y) \wedge (S(z))))$

$\Leftrightarrow \forall x \forall y \forall z (P(x) \rightarrow (Q(y) \wedge S(z)))$

b) 6 $\exists x \forall y (R(x,y) \rightarrow \exists z\, Q(x,z,y))$

$\Leftrightarrow \exists x \forall y (\neg R(x,y) \vee \exists z\, Q(x,z,y))$

$\Leftrightarrow \exists x \forall y \exists z (\neg R(x,y) \vee Q(x,z,y))$

$\Rightarrow \forall y \exists z (\neg R(a,y) \vee Q(a,z,y))$

$\Rightarrow \forall y (\neg R(a,y) \vee Q(a, f(y), y))$

$\Leftrightarrow \forall y (R(a,y) \rightarrow Q(a, f(y), y))$

c) Was this covered?

0 Does asking retherical questions yield any points

Will you have mercy on my soul?

Unfortunately I can't give you points for this.

B3 father(x): the father of x
mother(x): the mother of x
Younger(x, y): x is younger than y
jan: Jan
mieke: Mieke

10 a. $\forall x($ Younger(x, mother(jan)) $\rightarrow$ Younger(x, father(mieke)))

b. Younger(jan, mieke) $\wedge$
10 $\neg$ Younger(mother(father(jan)), father(mieke))

10 c. $\forall x$((mother(x) = mother(mieke)) $\rightarrow$ (father(x) = father(mieke)))

d. on backside

(ii)

	$(\neg A$	$\rightarrow$	$\neg C))$
	F	T	F
	F	T	T
T	F	T	F
T	F	T	T
F	T	F	F
T	T	T	T
T	T	F	F
T	T	T	T

B3 d.

10 $\forall x\,(\,\text{Younger}(x, \text{jan}) \to (\,\text{Younger}(x, \text{father}(\text{mother}(\text{mielie})))$
$\wedge\ \text{Younger}(x, \text{father}(\text{father}(\text{mielie}))))$

B4 a) Using M instead of fancy M

4.5

$\langle 2, 3 \rangle \notin M(R)$

$\langle h(y), h(x) \rangle \notin M(R)$

$\langle [\![y]\!]^M_h , [\![x]\!]^M_h \rangle \notin M(R)$

$M \nvDash R(y, x)[h]$ i

$2 \in M(Q)$

$[\![b]\!]^M_h \in M(Q)$

$M \vDash Q(b)[h]$ ii

from i and ii

$M \nvDash Q(b) \to R(y, x)[h]$

therefore $M \nvDash Q(c) \wedge (\,Q(b) \to R(y, x))[h]$

B4 b) $2 \in M(P)$

15 $h[x/2](x) \in M(P)$

$[\![x]\!]^{M}_{h[x/2]} \in M(P)$

$M \models P(x)\ [h[x/2]]$ i

$2 \in M(Q)$

$h[x/2](x) \in M(Q)$

$[\![x]\!]^{M}_{h[x/2]} \in M(Q)$

$M \models Q(x)\ [h[x/2]]$ ii

From i and ii

$M \models P(x) \wedge Q(x)\ [h[x/2]]$

therefore

$M \models \exists x (P(x) \wedge Q(x)) [h]$

c) v

To evaluate

$M \models \forall x \forall y (\neg Q(x) \rightarrow (R(x,y) \vee R(y,y)))[h]$

$1 \notin M(Q)$

~~h[x/1][y/1](x)~~ $h[x/1][y/1](x) \notin M(Q)$

~~[[x]]~~ $[[x]]^{M}_{h[x/1][y/1]} \notin M(Q)$

$M \not\models Q(x)[h[x/1][y/1]]$

$M \models \neg Q(x)[h[x/1][y/1]]$ iii

$\langle 1, 1 \rangle \notin M(R)$

$\langle h[x/1][y/1](x), h[x/1][y/1](x) \rangle \notin M(R)$

$\langle [[x]]^{M}_{h[x/1][y/1]}, [[y]]^{M}_{h[x/1][y/1]} \rangle \notin M(R)$

$M \not\models R(x,y)[h[x/1][y/1]]$ i

but also

$\langle h[x/1][y/1](y), h[x/1][y/1](y) \rangle \notin M(R)$

$\langle [[y]]^{M}_{h[x/1][y/1]}, [[y]]^{M}_{h[x/1][y/1]} \rangle \notin M(R)$

$M \not\models R(y,y)[h[x/1][y/1]]$ ii

from i and ii

$M \not\models R(x,y) \vee R(y,y)[h[x/1][y/1]]$ iv

From iii and iv

$$M \not\models \neg Q(x) \to (R(x,y) \vee R(y,y))[h[x/1][y/1]]$$

this means

$$M \not\models \exists y\, (\neg Q(x) \to (R(x,y) \vee R(y,y)))[h[x/1]]$$

which means

$$M \models \neg \forall y\, (\neg Q(x) \to (R(x,y) \vee R(y,y)))[h[x/1]]$$

$$M \not\models \forall y\, (\neg Q(x) \to (R(x,y) \vee R(y,y)))[h[x/1]]$$

therefore

$$M \not\models \exists x \forall y\, (\neg Q(x) \to (R(x,y) \vee R(y,y)))[h]$$

$$M \models \neg \forall x \forall y\, (\neg Q(x) \to (R(x,y) \vee R(y,y)))[h]$$

and thus

$$M \not\models \forall x \forall y\, (\neg Q(x) \to (R(x,y) \vee R(y,y)))[h]$$